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A HOMOTOPY METHOD APPLIED TO ELASTICA PROBLEMS

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Abstract—The inverse problem in nonlinear (incompressible) elastica theory, where the end positions and inclinations rather than the forces and moment are specified, is considered. Based on the globally convergent Chow–Yorke algorithm, a new homotopy method which is simple, accurate, stable, and efficient is developed. For comparison, numerical results of some other simple approaches (e.g. Newton's method based on shooting or finite differences, standard embedding) are presented. The new homotopy method does not require a good initial estimate, and is guaranteed to have no singular points. The homotopy method is applied to the problem of a circular elastica ring subjected to N symmetrical point loads, and numerical results are given for $N = 2, 3, 4$.

INTRODUCTION

The study of large, nonlinear deflections of rods or beams is important in many engineering problems, for example, frame structures [1, 2], leaf springs [3], cloth fabrics [4] and flexible linkages [5]. The theory for the nonlinear bending of thin rods, or elastica theory, was first formulated by Euler. In his *De Curvis Elasticis* [6] Euler stated that the curvature of a thin rod at any point is proportional to the local moment applied. The elastica problem was subsequently studied by many authors, including a noteworthy book [7] by Frisch-Fay in 1962. Only incompressible elastica are considered here.

Figure 1 shows an elastica subjected to terminal loads P', Q' and moment M'. The equation governing its shape is

$$EI\frac{d\theta}{ds'} = M' + Q'x' - P'y'. \tag{1}$$

Here EI is the flexural rigidity, θ is the local angle of inclination, and s' is the arc length along

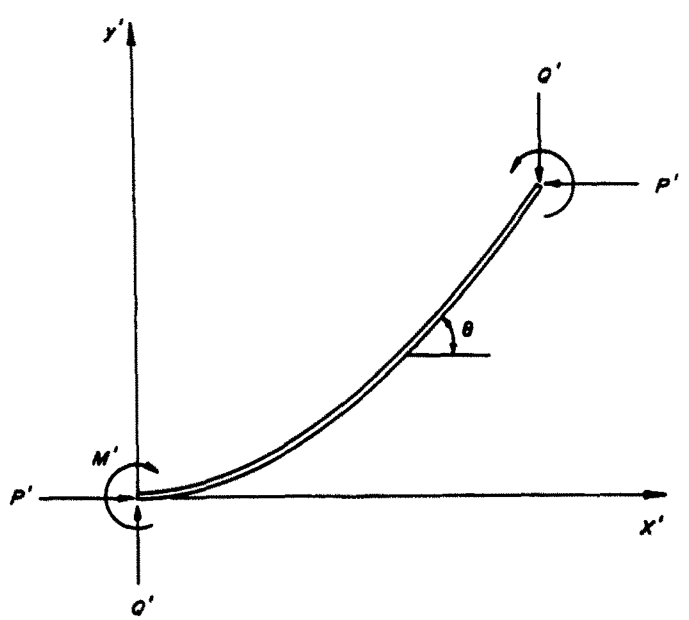

Fig. 1. The coordinate system.

the curve. Thus, $d\theta/ds'$ is the local curvature. Since

$$\frac{dx'}{ds'} = \cos\theta \tag{2}$$

$$\frac{dy'}{ds'} = \sin\theta. \tag{3}$$

Equation (1) can be written as

$$EI\frac{d^2\theta}{ds'^2} = Q'\cos\theta - P'\sin\theta. \tag{4}$$

If the loads and moment are known *a priori*, the solution to eqn (4) can be expressed as elliptic functions[7]. Although the solution is analytic, the process is extremely tedious and the accuracy is limited to the accuracy of the tables of elliptic functions. With the advent of the computer, eqn (4) can now be easily integrated numerically as an initial value problem with the initial conditions

$$s' = 0,\quad \frac{d\theta}{ds'} = \frac{M'}{EI},\ \theta = 0. \tag{5}$$

Using this method, one obtains the local inclination θ as a function of s'. The Cartesian positions x', y' can be found by integrating eqn (2) and (3).

The problem becomes extremely difficult even for numerical integration when the loads and moment P', Q', M' are unknowns while the end positions and inclinations are given. The usual method is to use shooting with Newton-Raphson iteration[8–12]. However, as pointed out by Shoup[10, 11], Newton's method would not converge unless the initial guess is extremely close to the correct solution. This is due to the fact that the elastica problem is very sensitive to end conditions, especially for the more complicated shapes. Finite element methods have also been applied to the elastica problem[13–16]. Since a functional form is assumed for each element, finite element solutions are approximate at best. Similar to the shooting method, a good initial guess is required for convergence. In addition, the finite element program is extremely tedious to write. In the next section, we shall investigate some of the simpler approaches to the numerical solution of the elastica problem.

2. RESULTS OF VARIOUS NUMERICAL APPROACHES

Let us nondimensionalize our variables as follows:

$$x = x'/L,\ y = y'/L,\ t = s'/L \tag{6}$$

$$M = LM'/EI,\ P = L^2P'/EI,\ Q = L^2Q'/EI. \tag{7}$$

Here L is the total length of the elastica. Then eqns (1)–(3) become

$$\frac{d\theta}{dt} = Qx - Py + M \tag{8}$$

$$\frac{dx}{dt} = \cos\theta \tag{9}$$

$$\frac{dy}{dt} = \sin\theta. \tag{10}$$

The boundary conditions are

$$x(0) = y(0) = \theta(0) = 0 \tag{11}$$

$$x(1) = a,\ y(1) = b,\ \theta(1) = c. \tag{12}$$

There are six conditions and six unknowns: x, y, θ, Q, P, M.

2.1. *Newton's method—shooting*

Newton's method based on shooting is the simplest approach. Let $v = (Q, P, M)$, and denote the solution to the initial value problem eqns (8)–(11) by $x(t; v)$, $y(t; v)$, $\theta(t; v)$. Then clearly eqns (8)–(12) is equivalent to

$$F(v)=\begin{Bmatrix} x(1;v)-a \\ y(1;v)-b \\ \theta(1;v)-c \end{Bmatrix}=0. \qquad (13)$$

Newton's method requires the Jacobian matrix $DF(v)$, but that can be computed accurately with no difficulty here. Partials like $\partial x(1)/\partial Q$ can be easily computed by numerically solving a larger nonlinear initial value problem as described in [24] and for this problem $\partial F/\partial v_i$ is only slightly more expensive to obtain than $F(v)$ itself.

For $a=0$, $b=2/\pi$, $c=\pi$, the exact solution is $\bar{v}=(Q,P,M)=(0,0,\pi)$. Starting at $v=(0,0,1.85)$, Newton's method on eqn (13) failed to coverge, and this was typical behavior for other boundary conditions and other starting points.

$F(v)$ in eqn (13) is in effect computed by an integration (integrating an ordinary differential equation). An approach based directly on numerical integration (quadrature) follows. The first integral of eqns (8) and (11) gives

$$H(\theta,Q,P,M)=[M^2+2Q\sin\theta+2P(\cos\theta-1)]^{-(1/2)}=1/\dot{\theta}.$$

Equations (9), (10) and (12) then yield the nonlinear system of equations

$$\begin{aligned}&\int_0^c H(\theta,Q,P,M)\cos\theta\,d\theta-a=0\\ &\int_0^c H(\theta,Q,P,M,)\sin\theta\,d\theta-b=0\\ &\int_0^c H(\theta,Q,P,M)\,d\theta-1=0\end{aligned} \qquad (13A)$$

equivalent to eqns (8)–(12). H is not defined for all Q, P, M, which means a poor initial guess may result in H becoming undefined. Even at the solution (Q, P, M), H may not be analytic on the closed interval $[0, c]$ (which happens if the rod is bell-shaped, for example). This latter fact means that only an adaptive quadrature algorithm will be accurate, and such quadrature is very expensive, especially if c is large. Furthermore, eqn (13A) cannot handle the important cases with $c=0$. For these reasons, eqn (13A) is not competitive with eqn (13). Using the same data as above, Newton's method on eqn (13A) also failed.

There are many related locally convergent methods such as n-dimensional *regula falsi* [25], n-dimensional secant, nonlinear Gauss–Siedel, Brown's method, Broyden's method and the quasi-Newton BFGS method. None of these methods are truly *globally convergent* and their domain of convergence is fairly small for the elastica problems considered here. To illustrate, the widely used subrouting ZSYSTM (based on Brown's method) from the IMSL package diverged for $a=0$, $b=0.9$, $c=\pi$ starting from the solution for $a=0$, $b=0.8$, $c=\pi$, and this was typical behavior. The very sophisticated quasi-newton code HYBRJ, developed at Argonne National Laboratory [26], also failed in the above situation. However, HYBRJ did converge in about 40% of the cases tried, which was far better than any other locally convergent method.

2.2. *Newton's method—finite difference approximation*

Equation (8) is equivalent to

$$\ddot{\theta}(t)=Q\cos\theta(t)-P\sin\theta(t), \qquad (14)$$

which eliminates the constant M. Let $h=1/(n+1)$, $t_i=ih$, $i=0,\ldots,n+1$, and X_i, Y_i, θ_i be approximations to $x(t_i)$, $y(t_i)$, $\theta(t_i)$ respectively. Using a second order accurate finite difference approximation to eqns (9)–(12), (14) yields the equations

$$\begin{pmatrix} 0 & -1 & 0 & & \\ 1 & 0 & -1 & \bigcirc & \\ 0 & 1 & 0 & & \\ & & & \ddots & \\ & \bigcirc & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdot \\ \cdot \\ X_{n-1} \\ X_n \end{bmatrix}+\begin{bmatrix} 2h\cos\theta_1 \\ 2h\cos\theta_2 \\ 2h\cos\theta_3 \\ \cdot \\ \cdot \\ 2h\cos\theta_{n-1} \\ 2h\cos\theta_n-a \end{bmatrix}=0$$

$$\begin{pmatrix} 0 & -1 & 0 & & & \\ 1 & 0 & -1 & & \bigcirc & \\ 0 & 1 & 0 & & & \\ & & & \ddots & & \\ & \bigcirc & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_{n-1} \\ Y_n \end{bmatrix} + \begin{bmatrix} 2h\sin\theta_1 \\ 2h\sin\theta_2 \\ 2h\sin\theta_3 \\ \vdots \\ 2h\sin\theta_{n-1} \\ 2h\sin\theta_n - b \end{bmatrix} = 0$$

$$\begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & \bigcirc & \\ 0 & -1 & 2 & & & \\ & & & \ddots & & \\ & \bigcirc & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{bmatrix} + \begin{bmatrix} h^2(Q\cos\theta_1 - P\sin\theta_1) \\ h^2(Q\cos\theta_2 - P\sin\theta_2) \\ \vdots \\ h^2(Q\cos\theta_{n-1} - P\sin\theta_{n-1}) \\ h^2(Q\cos\theta_n - P\sin\theta_n) - c \end{bmatrix} = 0$$

$$5\theta_1 - 4\theta_2 + \theta_3 + h^2 Q = 0$$
$$5\theta_n - 4\theta_{n-1} + \theta_{n-2} - 2c + h^2(Q\cos c - P\sin c) = 0$$

which can be written as

$$G(z) = 0, \tag{15}$$

where

$$z = (X_1, \ldots, X_n, Y_1, \ldots, Y_n, \theta_1, \ldots, \theta_n, P, Q).$$

Solving eqn (15) by Newton's method is even more difficult than solving eqn (13), because the entire shape of the rod as well as P and Q must be estimated in the initial guess. For the solution to eqn (15) to be a reasonably accurate approximation to the true continuous solution, n must be at least 9, which means eqn (15) must be at least 29 dimensional. For $a = 0$, $b = 2/\pi$, $c = \pi$, $n = 9$, starting at $z = 0$ Newton's method failed. The size of eqn (15) coupled with the burden of finding a good initial z make solving (15) by Newton's method impractical.

2.3. *Imbedding—shooting*

A more sophisticated approach is to consider the family of problems

$$\psi_w(\lambda, v) = \lambda F(v) + (1-\lambda)(v - w) = 0 \tag{16}$$

for $0 \le \lambda \le 1$, where $w \in E^3$ is fixed and $F(v)$ is given by eqn (13). The imbedding algorithm is to increase λ from 0 to 1 and track the solutions of $\psi_w = 0$ from w (at $\lambda = 0$) to the solution $\bar{v}$ of $F(v) = 0$ (at $\lambda = 1$). This approach fails if $\psi_w(\bar{\lambda}, v) = 0$ has no solution or if the Jacobian matrix $D_v\psi_w(\bar{\lambda}, v)$ is singular, because then the solutions cannot be "continued" beyond $\lambda = \bar{\lambda}$.

The exact solution for $a = 0.98584269$, $b = 0.14607461$, $c = -0.06339365$ is $\bar{v} = (-2, 1, 1)$. Starting at $w = (-3, 1.5, 1.5)$ the Jacobian matrix $D_v\psi_w(\lambda, v)$ becomes singular at $\lambda = 0.12$ and therefore imbedding fails. For some problems changing the sign of some components of $F(v)$ helps, but imbedding failed for every problem of the form $\operatorname{diag}(\pm 1, \pm 1, \pm 1)\, F(v)$.

2.4. *Imbedding—finite difference approximation*

The imbedding family here is

$$\Omega_w(\lambda, z) = \lambda G(z) + (1-\lambda)(z - w) = 0, \tag{17}$$

where $G(z)$ are the finite difference equations given by eqn (15) and $w \in E^{3n+2}$. For $a = 0$, $b = 2/\pi$, $c = \pi$ and $n = 9$, the exact solution is

$$\bar{z} = \left(\frac{\sin 0.1\pi}{\pi}, \ldots, \frac{\sin 0.9\pi}{\pi}, \frac{1-\cos 0.1\pi}{\pi}, \ldots, \frac{1-\cos 0.9\pi}{\pi}, 0.1\pi, \ldots, 0.9\pi, 0, 0\right).$$

For the starting point $w = 0$, the Jacobian matrix $D_v\Omega_w(\lambda, z)$ becomes singular at $\lambda = 0.99925$, and imbedding again fails. For some boundary conditions and starting points, $\|z\| \to \infty$ as $\lambda \to 1$.

2.5. *Chow–Yorke algorithm—shooting*

The Chow-Yorke algorithm is a homotopy method using the same homotopy map as eqn (16), but differs from standard imbedding in several important respects. It is globally convergent with probability one for certain classes of problems[17–19] and is unaffected by "singular points". The computer implementation of the Chow-Yorke algorithm is very different from that of the typical imbedding algorithm. See Watson[19] for the theoretical background and details of the computer implementation. Basically the supporting theory says that for almost all w, there is a zero curve γ of $\psi_w(\lambda, v)$, emanating from $(0, w)$, along which the Jacobian matrix $D\psi_w$ (with respect to both λ *and* v) has full rank. γ either reaches a zero $\bar{v}$ of F (at $\lambda = 1$) or wanders off to infinity. The Chow-Yorke algorithm is to track γ, where λ and v are both *dependent* variables along γ.

Using the same boundary conditions and starting point as in Section 2.3, the Chow-Yorke algorithm also failed. γ turns back at $\lambda = 0.12$ and goes to infinity with $\lambda \to 0$, $\|v\| \to \infty$.

2.6. *Chow–Yorke algorithm—finite difference approximation*

The theory is the same as in Section 2.5, except the homotopy map eqn (17) is used. Using the same boundary conditions and starting point as in Section 2.4, the Chow-Yorke algorithm again failed. The zero curve γ goes off to infinity ($\|z\| \to \infty$) and either $\lambda \to 0$ or $\lambda \to 1$, depending on the problem and starting point. The Chow-Yorke algorithm does converge for $n = 4$ ($h = 0.2$),but the mesh is too coarse for the solution to be of much value.

2.7. *A homotopy method based on Chow–Yorke algorithm*

Let $v = (Q, P, M)$, $w = (w_1, w_2, w_3)$, and $x(t; v)$, $y(t; v)$, $\theta(t; v)$ be the same as in Section 2.1. Now define ρ_w:$[0, 1)xE^3 \to E^3$ by

$$\rho_w(\lambda, v) = \rho(w, \lambda, v) = \begin{Bmatrix} x(1; v) - [\lambda a + (1-\lambda)w_1] \\ y(1; v) - [\lambda b + (1-\lambda)w_2] \\ \theta(1; v) - [\lambda c + (1-\lambda)w_3] \end{Bmatrix} \tag{18}$$

The Chow-Yorke algorithm is based on the following fact[19]:

Lemma. Suppose that the Jacobian matrix $D\rho(w, \lambda, v)$ of ρ has full rank on $\rho^{-1}(0)$. Then for almost all $w \in E^3$ (in the sense of Lebesgue measure), the Jacobian matrix $D\rho_w(\lambda, v)$ of ρ_W also has full rank on $\rho_w^{-1}(0)$.

The implication of this Lemma is that the set of zeros of ρ_w in $[0,1) \times E^3$ consists of smooth, disjoint curves whose only endpoints must lie in $\{0\} \times E^3$ or $\{1\} \times E^3$. Furthermore the Jacobian matrix $D\rho_w$ has full rank along these curves. Thus if

$$\rho_w(0, \bar{v}) = 0$$

there exists a smooth curve γ, emanating from $(0, \bar{v})$, along which the Jacobian matrix $D\rho_w$ has full rank. γ either reaches $\lambda = 1$, or wanders off to infinity. The above statements hold with probability one, in the sense of holding for almost all w.

The proposed homotopy method is as follows: Choose a pair w, $\bar{v}$ such that $\rho_w(0, \bar{v}) = 0$; this is easily done. Then track the zero curve γ of ρ_w emanating from $(0, \bar{v})$ until $\lambda = 1$. If $\rho_W(1, \bar{v}) = 0$, then from eqn (18) $\bar{v}$ solves the boundary value problem eqns (8)–(12). The zero curve γ is the trajectory of the initial value problem

$$\frac{d}{ds}\rho_w(\lambda(s), v(s)) = \left[\frac{\partial\rho_w(\lambda, v)}{\partial\lambda}, D_v\rho_w(\lambda, v)\right]\begin{Bmatrix} \frac{d\lambda}{ds} \\ \frac{dv}{ds} \end{Bmatrix} = 0 \tag{19}$$

$$\left\|\left(\frac{d\lambda}{ds}, \frac{dv}{ds}\right)\right\|_2 = 0 \tag{20}$$

$$\lambda(0) = 0, \; v(0) = \bar{v} \tag{21}$$

where s is the arc length along γ. Equations (19)–(21) are best solved by a variable step, variable order Adams method as described in Shampine[20]. Note that the derivative $(d\lambda/ds, dv/ds)$ is only implicitly defined by eqns (19)–(20), and some nontrivial numerical linear algebra is required for its calculation. The details of calculating $(d\lambda/ds, dv/ds)$ and for solving eqns (19)–(21) are similar to those of the fixed point algorithm in Watson[19].

The Jacobian matrix $D_v\rho_w(\lambda, v)$ involves partials like $\partial x(1)/\partial Q$. These could be approximated by finite differences, but the following procedure is more accurate and efficient. Let

$$u(t) = (x(t), y(t), \theta(t), \partial x(t)/\partial Q, \partial y(t)/\partial Q, \partial\theta(t)/\partial Q).$$

Then the solution to the initial value problem

$$\begin{aligned} \dot{u}_1 &= \cos u_3 \\ \dot{u}_2 &= \sin u_3 \\ \dot{u}_3 &= Qu_1 - Pu_2 + M \\ \dot{u}_4 &= -u_6 \sin u_3 \\ \dot{u}_5 &= u_6 \cos u_3 \\ \dot{u}_6 &= Qu_4 + u_1 - Pu_5 \\ u(0) &= 0 \end{aligned} \tag{22}$$

gives, e.g. $u_4(1) = \partial x(1)/\partial Q$. Similarly for P and M.

An important point is that the homotopy map eqn (18) is *not* just an imbedding. λ does not have to increase monotonically from 0 to 1 along γ and there are never any "singular points" along γ. Because of the full rank of the Jacobian matrix and the way in which γ is tracked, "turning points" pose no difficulties whatsoever.

In the next section this homotopy method based on the Chow–Yorke algorithm is applied to a practical problem in elastica theory.

3. THE CIRCULAR ELASTICA RING SUBJECTED TO SYMMETRIC POINT LOADS

3.1. *Two loads*

Consider a naturally straight elastica rod of length $2L$ bent into a circular ring. Figure 2 shows such a ring subjected to two symmetric point loads. This problem has been studied previously by several authors[21–23] by the analytical method using elliptic functions. There are two disadvantages using this method. Firstly the accuracy of the results cannot be better than the accuracy of the elliptic functions used. Secondly, the variables b, Q, M are determined *aposteriori*, i.e. cannot be controlled. Table 1(a) shows typical results obtained by this method published by Frisch-Fay[7]. Also shown in Table 1(a) are those obtained by our homotopy algorithm. All our figures are correct while traditional method gives only two figure accuracy. Since we can control our parameters, Table 1(b) shows the results for given b, using the present method.

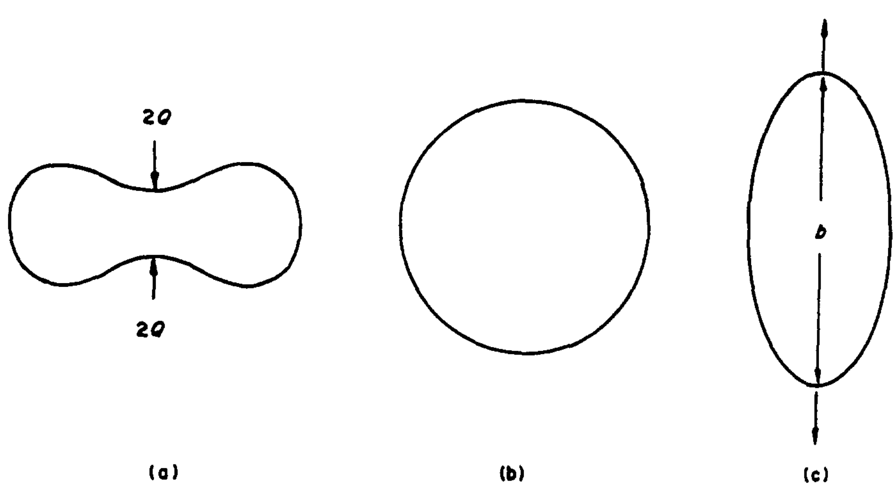

Fig. 2. Deformation of a ring by two loads (a) compression, (b) free, (c) extension.

Table 1. Deformation of a ring by two loads

Table 1(a).

	Q		M	
b	Ref.[7]	Present method	Ref.[7]	Present method
0.000000	39.68	39.650635	—	−6.055263
0.239369	25.66	25.619315	−2.985	−2.986823
0.457730	13.72	13.703192	0.000	0.011793
0.709857	−9.475	−9.429116	4.891	4.888416
0.806597	−34.54	−34.359221	8.366	8.354052

Table 1(b).

b	Q	M
0	39.65063	−6.055263
0.1	33.33404	−4.766169
0.2	27.72253	−3.492393
0.3	22.42202	−2.195502
0.4	17.04025	−.830645
0.5	11.05332	0.665996
0.6	3.515156	2.404911
$2/\pi$	0	π
0.7	−7.894501	4.625114
0.8	−31.70085	8.040393
0.9	−137.2019	16.56551
1.0	$-\infty$	∞

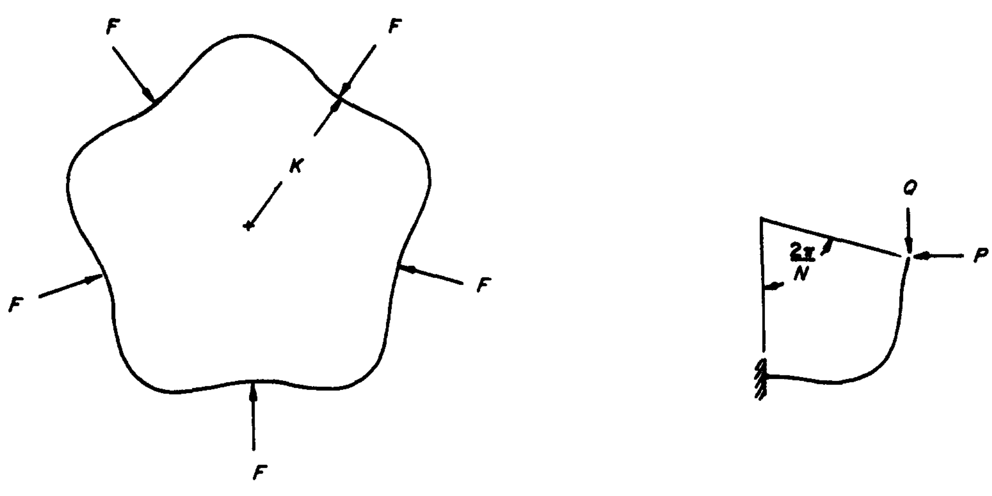

Fig. 3. Schematic diagram for N symmetric loads.

In what follows, we shall generate the results for the deformation of a circular elastica ring by more than two symmetric loads. To the authors' knowledge these results have never been published before.

3.2. *More than two loads*

Figure 3 shows a circular elastica ring of free radius R subjected to N symmetrical loads F' Set $L = 2\pi R/N$ and normalize all lengths, forces, moments as in eqns (6)–(7). The problem then reduces to eqns (8)–(12) with

$$a = K\sin\frac{2\pi}{N},\ b = K\left(1-\cos\frac{2\pi}{N}\right),\ c = \frac{2\pi}{N} \tag{23}$$

where K is the distance from the center of symmetry to the point of load application. For given K one can integrate for the forces and moment Q, P, and M. Considering one segment of the ring $0 \le \theta \le 2\pi/N$, the normalized force F can be calculated as follows

$$\frac{L^2F'}{EI} = F = 2(Q^2+P^2)^{1/2}\, b(a^2+b^2)^{-(1/2)} \tag{24}$$

Using symmetry which dictates $Pb = Qa$ one obtains the simple result

$$F = 2Q. \tag{25}$$

The numerical values for force F as a function of distance K for $N = 3$ and $N = 4$ are tabulated on Tables 2 and 3. Figure 4 shows some of the deformation configurations for four loads. It is obvious that the complicated geometries are unsuitable for a finite element formulation.

Table 2. Deformation of a ring by three loads

K	F	M
0	93.25454	−10.381892
0.05	82.67937	−9.199388
0.10	73.63733	−8.071149
0.15	65.63493	−6.971627
0.20	58.28358	−5.877444
0.25	51.22793	−4.763989
0.30	44.06862	−3.601012
0.35	36.23102	−2.344733
0.40	26.63785	−0.919440
0.45	12.56279	0.8390613
$3/2\pi$	0	$2\pi/3$
0.50	−16.93792	3.448575
$1/\sqrt{3}$	$-\infty$	∞

Table 3. Deformation of a ring by four loads

K	F	M
0	86.80169	−12.355771
0.05	79.07513	−11.346689
0.10	72.41183	−10.389538
0.15	66.54758	−9.468586
0.20	61.28057	−8.570414
0.25	56.44677	−7.682686
0.30	51.90093	−6.793016
0.35	47.49829	−5.887680
0.40	43.07053	−4.949739
0.45	38.38345	−3.955757
0.50	33.04092	−2.868894
0.55	26.20991	−1.621358
0.60	15.53799	−0.055590
$2/\pi$	0	$\pi/2$
0.65	−10.66333	2.407977
0.68	−88.27135	6.051593
$1/\sqrt{2}$	$-\infty$	∞

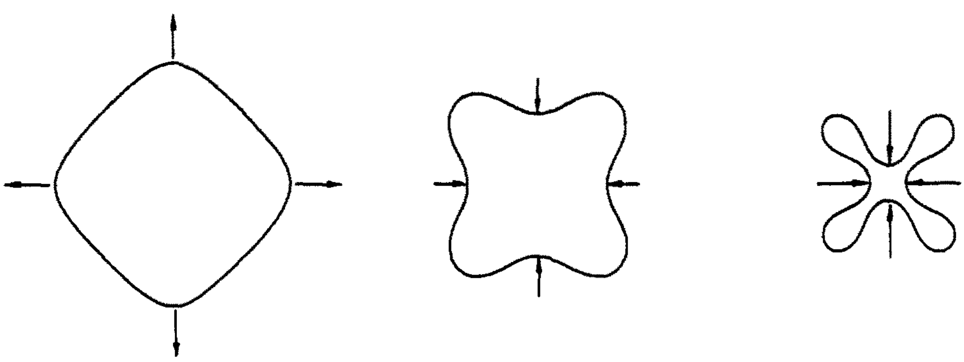

Fig. 4. Integrated deformation shapes for 4 symmetric loads (a) $K = 0.68$, (b) $K = 0.4$, (c) $K = 0.1$.

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